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STABLE EQUILIBRIA WITH VARIABLE DIFFUSION

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For a scalar nonlinear parabolic equation in one space dimension with homogeneous Neumann boundary conditions, criteria are given on the diffusion coefficient to ensure that the stable equilibrium solutions are constant functions regardless of the nonlinearities. The Dirichlet problem is also discussed.

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Stable equilibria with variable diffusion

by

Michel Chipot and Jack K. Hale

ABSTRACT

For a scalar nonlinear parabolic equation in one space dimension with homogeneous Neumann boundary conditions, criteria are given on the diffusion coefficient to ensure that the stable equilibrium solutions are constant functions regardless of the nonlinearities. The Dirichlet problem is also discussed.

Consider the equation

$$u_t = (au_x)_x + f(u), \qquad 0 < x < 1, \tag{1}$$

with homogeneous Neumann conditions

$$u_x = 0 \text{ at } x = 0, \; x = 1, \tag{2}$$

where $a \in C^2[0,1]$ is a positive function and $f \in C^2(\mathbb{R})$. System (1), (2) defines a local dynamical system in $H^1(0,1) = W^{1,2}(0,1)$ (see Henry [8]). Furthermore, the ω-limit set of any bounded orbit is exactly one equilibrium point; that is, a solution of the equations

$$(au_x)_x + f(u) = 0, \qquad 0 < x < 1 \tag{3}$$

$$u_x = 0 \text{ at } x = 0, \; x = 1 \tag{4}$$

(see Matano [11], Hale and Massatt [6], Zelenyak [15]).

The first result is the following.

Theorem 1. *If* $a''(x) \le 0$ *on* $[0,1]$, *then every nonconstant equilibrium solution of* (1),(2) *is unstable*.

For $a(x) = c > 0$, where c is a constant, this result was proved by Chafee [2]. Note that the conclusion in the theorem is valid for every function f. Matano [13] has given an example (1),(2) for which there are stable non-constant equilibrium solutions. The function f can be chosen to be a cubic with three simple zeros and the function a is ≥ 1 on intervals $[0,\alpha]$, $[\beta,1]$ and $\le \varepsilon$ on $[\gamma,\delta]$, $\alpha < \gamma < \delta < \beta$, and ε is sufficiently small. Numerical examples indicated the same property can occur with $a \ge 1$ on $[0,\alpha]$ and $\le \varepsilon$ on

$[\beta,1]$ with $\alpha < \beta$. These examples show clearly that more work is needed to determine the optimal conditions on a for the conclusions of the theorem to hold.

For a more general parabolic equation in several space dimensions in a bounded domain Ω and the diffusion coefficients constant, Casten and Holland [1], Matano [12] have shown that Theorem 1 is true if Ω is convex. The authors have had no success in extending this result in a significant way with variable diffusion coefficients. If Ω is not convex and the diffusion coefficients are constants, nonconstant equilibrium solutions can occur (see Matano [12], Hale and Vegas [7], Vegas [14]) for certain nonlinear functions f. If the diffusion coefficient in (1) is constant and f is a function of $u(\cdot,t)$, for example, $f(u(x,t)) = g(u(x,t), \int_0^1 \alpha(x)u(x,t)dx)$, then stable nonequilibrium solutions can occur (see Chafee [3]). A similar situation was considered in several space dimensions and nonconvex domains by Keyfitz and Kuiper [9]. For constant diffusion in (1) and f replaced by $s(x)f(u)$, Fleming [5], (see also Henry [8]) has considered how the existence of stable nonconstant solutions depend on $s(x)$. All of these papers should be reconsidered with variable diffusion.

There is an analogue of Theorem 1 for the Dirichlet problem.

<u>Theorem 2.</u> <u>Consider the equation</u> (1) <u>with Dirichlet boundary conditions</u>

$$u = 0 \quad \text{at} \quad x = 0, \quad x = 1.$$

<u>If</u> $a''(x) \le 0$ <u>on</u> $[0,1]$ <u>and</u> $v(x)$ <u>is a nonconstant equilibrium solution such that</u> $v_x = 0$ <u>at two points in</u> $(0,1)$, <u>then</u> v <u>is unstable</u>.

For $a(x) = c > 0$, this result was proved by Chafee and Infante [4], Maginu [10].

<u>Proof of Theorem 1.</u> If v is a solution of (3),(4) and

$$\mathscr{A}(\phi) = \int_0^1 [a\phi_x^2 - f'(v)\phi^2]dx,$$

then the first eigenvalue λ_1 of the operator

$$Lu = -(au_x)_x - f'(v)u$$

is given by

$$\lambda_1 = \min\{\mathscr{A}(\phi) : \phi \in H^1(0,1), |\phi|_{L^2(0,1)} = 1\}.$$

Furthermore, the equilibrium solution v of (1),(2) is unstable if $\lambda_1 < 0$ (see, for example, Henry [8]). Consequently, it is sufficient to show that the hypotheses of the theorem imply $\lambda_1 < 0$ if v is not a constant and this will be the case if we show that $\mathscr{A}(av_x) < 0$ if v is not a constant. We have

$$\begin{aligned}
\mathscr{A}(av_x) &= \int_0^1 [a[(av_x)_x]^2 - f'(v)(av_x)^2]dx \\
&= \int_0^1 af^2(v)dx - \int_0^1 f(v)_x a\, av_x dx \\
&= \int_0^1 af^2(v)dx + \int_0^1 f(v)a(av_x)_x dx + \int_0^1 f(v)a'av_x dx \\
&= -\int_0^1 (av_x)_x a'(av_x)dx
\end{aligned}$$

To compute the last expression, we integrate by parts to obtain

$$\int_0^1 (av_x)_x a'(av_x)dx = -\int_0^1 av_x a'' av_x dx - \int_0^1 av_x a'(av_x)_x dx$$

and, thus,

$$\int_0^1 (av_x)_x a'(av_x)dx = -\frac{1}{2}\int_0^1 a''(av_x)^2 dx.$$

Therefore,

$$\mathcal{A}(av_x) = \frac{1}{2}\int_0^1 a''(av_x)^2 dx \leq 0 \tag{5}$$

since we have assumed $a'' \leq 0$.

If $\mathcal{A}(av_x) < 0$, then $\lambda_1 < 0$ and the solution v is unstable. Thus, suppose $\lambda_1 \geq 0$. From (5), this implies $\mathcal{A}(av_x) = 0$ which implies $\lambda_1 = 0$. Since the first eigenvalue of L is simple, there is a ϕ satisfying $L\phi = 0$, $\phi_x = 0$ at $x = 0$, $x = 1$, $|\phi|_{L^2(0,1)} = 1$. Also, $\mathcal{A}(av_x) = 0$ implies there is a constant c such that $av_x = c\phi$ on $[0,1]$. If $c \neq 0$, then $v_x = 0$ at $x = 0$, $x = 1$ and $a > 0$ imply $\phi = 0$ at $x = 0$, $x = 1$. But this would imply $\phi = 0$ on $[0,1]$ which is a contradiction. Thus, $c = 0$ and $av_x = 0$, $v_x = 0$ and $v =$ constant. This proves the theorem.

Proof of Theorem 2. Suppose v is an equilibrium solution, $v_x = 0$ at $x = \alpha$, $x = \beta$, $\alpha,\beta \in (0,1)$. Then v is an equilibrium solution of the Neumann problem on the interval $[\alpha,\beta]$. As in the proof of Theorem 1, one shows that the first eigenvalue λ_1 of the linear variational operator is negative if v is not a constant. By the characterization of λ_1 as a minimum of the functional $\mathcal{A}(u)$, it follows that λ_1 for the Dirichlet problem on $[0,1]$ is negative. This proves the theorem.

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